Isothermal coordinates on singular minimal surfaces (Problems in the Calculus of Variations and Related Topics)

Yamada, Sumio

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Sumio Yamada
Mathematical Institute
Tohoku University
yamada@math.tohoku.ac.jp

1 Introduction

J.Taylor's list ([T]) of configurations for so-called $(M,0,\delta)$-minimal sets ([Alm]) in $\mathbf{R}^3$ include a singularity type where three minimal surfaces are meeting along a real analytic ([KNS]) singular curve with 120° degree angle. In this article we introduce a local conformal parametrization of such a configuration by a 2-dimensional simplicial complex Y_0 consisting of three half discs whose diameters are identified to form a 1-dimensional face. The parametrization functions as an isothermal coordinate system of the neighborhood of the singular surface.

We introduce two different methods in order to construct such parametrizations, both utilizing the real analyticity of the singular surface, and the Euclidean ambient geometry. What is required to establish a conformal parametrization is a Beltrami equation locally defined on a half plane.

Then we point out that the conformal parameterization from the 2-dimensional simplicial complex into the singular surface with the three balanced surfaces has a mean value property. There has been much work on the subject of harmonic analysis on Euclidean buildings where harmonic functions are defined on the buildings. The conformal harmonic parameterization of the singular minimal surfaces can be regarded as a graph of a harmonic function defined over the simplcial complex Y_0.

The content of this article is a part of an ongoing project(cf. [MY],[MY2]) by C.Mese and the author.

2 Identifying Beltrami equation

Let

$$\triangle^+ = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < 1, y > 0\}$$

and consider three copies of $\triangle^+$ and label them $\triangle_1^+, \triangle_2^+, \triangle_3^+$ to distinguish one from another. Let

$$A_i = \{(x,y) \in \mathbf{R}^2 : -1 < x < 1, y = 0\} \subset \overline{\triangle_i^+}.$$

Identify the points of A_i and A_j by the identity map $\mathrm{Id} : A_i \to A_j$ and Y_0 be the union of the 3 half-discs $\triangle_i$ with this identification on A_i's and denote the A_i's by A. For a map α from domain Y_0, we will denote the restriction of α to $\triangle_i^+$ by α_i and write $\alpha = (\alpha_1, \alpha_2, \alpha_n)$.

Lemma 1 *Let $\Gamma \subset \mathbf{R}^3$ be a real-analytic curve and $\Sigma_i \subset \mathbf{R}^3$ $(i = 1, 2, 3)$ be a real-analytic surface so that $\Sigma_i \cup \Gamma$ is a real-analytic surface with boundary Γ. We assume further that the three surfaces are balanced, namely meeting along Γ at $120°$ degree angle. Then for every $p_0 \in \Gamma$, there exists a real-analytic map $u_i : \triangle^+ \cup A \to \Sigma_i \cup \Gamma$ so that $u(\triangle^+) \subset \Sigma_i$ and $p_0 \in u_i(A) \subset \Gamma$. Furthermore, $u_i|_A$ is a constant speed parametrization of Γ in a neighborhood of p_0, with the constant speed shared by $u_1|_A, u_2|_A$ and $u_3|_A$.*

PROOF.[**Construction 1**] We will present a version of the proof where the regularity is optimal. Then the statement for the real analytic data follows from the stronger statement. Let Σ be Σ_1 for now. Let $\gamma : (-t_0, t_0) \to \Gamma$ be an arclength parametrization of Γ so that $\gamma(0) = p_0$. Then $t \in (-t_0, t_0) \mapsto \gamma'(t)$ is a C^1 map. Let $P(t) \subset \mathbf{R}^N$ be the hyperplane containing the point $\gamma(t)$ and perpendicular to $\gamma'(t)$. Because Γ is C^2, for a sufficiently small neighborhood $\mathcal{V}$ of p_0, every point $p \in \mathcal{V}$ belongs to a unique $P(t)$. The size of $\mathcal{V}$ is only dependent on the curvature of Γ. Define a map $\mathcal{P} : \mathcal{V} \subset \mathbf{R}^N \to \mathbf{R}^N$ so that $\mathcal{P}$ is the hyperplane $P(t)$ containing p. In other words, if $\pi_\Gamma : \mathcal{V} \to \Gamma$ is the nearest point projection map, then $\mathcal{P}(p) = P \circ \gamma^{-1} \circ \pi_\Gamma(p)$. Since Γ is a C^2 curve, π_Γ is C^1 ([Si] 2.12.3). Therefore, as a composition of C^1 maps, $\mathcal{P}$ is also C^1. Furthermore, since Σ is C^2, the map $\mathcal{T}$ which takes $p \in (\Sigma \cup \Gamma) \cap \mathcal{V}$ to the 2-plane tangent to Σ at p is C^1. Since $\mathcal{P}$ and $\mathcal{T}$ are C^1 maps, we can let $V : (\Sigma \cup \Gamma) \cap \mathcal{V} \to \mathbf{R}^N$ be the C^1 map so that $V(p)$ is the unit vector associated to the line $\mathcal{P}(p) \cap \mathcal{T}(p)$. Now define a C^1 map $H : (\Sigma \cup \Gamma) \cap \mathcal{V} \to \mathbf{R}^N$ by setting $H(p)$ to be the unit vector perpendicular to $V(p)$ in the plane $\mathcal{T}(p)$. Thus V and H are orthonormal vector fields on $(\Sigma \cap \Gamma) \cap \mathcal{V}$. Let $\sigma_t(s)$ be an arclength parametrization of the curve $(\Sigma \cap \Gamma) \cap P(t)$ with $\sigma_t(0) = \gamma(t)$. By construction $\sigma_t'(s) = V(\sigma_t(s))$, i.e. σ_t is a characteristic curve of the vector field V. We define $\gamma_s(t)$ as the characteristic curve of the vector field H with condition $\gamma_s(0) = \sigma_0(s)$. The existence and uniqueness of $\gamma_s(t)$ follows from the standard ODE theory because H is a C^1 vector field on $(\Sigma \cup \Gamma) \cap \mathcal{V}$ and $\gamma_0 = \gamma$ is the characteristic curve of H whose image is $\Gamma \cap \mathcal{V}$. In this way, we have constructed a pair of orthogonal foliations on a neighborhood $\mathcal{U} \subset \Sigma \cup \Gamma$ (with $\mathcal{U}$ chosen smaller than $\mathcal{V}$ if necessary) of p_0.

We define a map $\phi : \mathcal{U} \to \mathbf{R}^2$ as follows. For $p \in \mathcal{U}$, let the γ_s and σ_t be the curves intersecting at p. Then we set $\phi(p) = (t, s)$. Then the C^2 map ϕ^{-1} defines a parametrization of a neighborhood of p_0 by an open neighborhood of the upper half space of the ts-plane. Furthermore the pulled-back metric $(\phi^{-1})^* g_0$ of the 3-dimensional Euclidean metric g_0, which we denote by G is represented on the upper half ts-plane as a diagonal matrix near the origin, since the surface is orthogonally foliated by the leaves $\{\sigma_t\}$ and $\{\gamma_s\}$. We are done by choosing $r > 0$ sufficiently small and letting $u : \triangle^+ \cup A \to \Sigma \cup \Gamma$ be defined $u(x, y) = \phi^{-1}(rx, ry)$.

Repeat the same argument for $i = 2$ and 3. Q.E.D.

PROOF.[**Construction 2**] We use the so-called hodographic projection of [KNS] to parameterize a neighborhood of p_0. The surfaces Σ_2 and Σ_3 are locally graphs over the tangent plane $T_{p_0}\Sigma_1$. We suppose that Σ_3 lies above the plane, Σ_2 below. Let Π_2 and Π_3 be the orthogonal projection maps from $\Sigma_2 \cap B_\delta(p_0)$ and $\Sigma_3 \cap B_\delta(p_0)$ to $T_{p_0}\Sigma_1$ for sufficiently small δ. We can choose the coordinates so that $p_0 = (0,0,0) \in \mathbf{R}^3$, the tangent line to γ at p_0 is the x_1-axis and $T_{p_0}\Sigma_1$ is the x_1x_2-plane. Define u_2, u_3 by the conditions $\Pi_2^{-1}(x_1,x_2) = (x_1,x_2,u_2(x)) \in \Sigma_2$ and $\Pi_3^{-1}(x_1,x_2) = (x_1,x_2,u_3(x)) \in \Sigma_3$. Near the origin, the map $h(x_1,x_2) := (x_1, u_3(x) - u_2(x))$ is of rank two. It sends $\Pi_2(\gamma) = \Pi_3(\gamma)$ to the x_1-axis and its image is contained in the upper half plane. The map $h \circ \Pi_2$ is the hodographic projection and its inverse map $\Pi_2^{-1} \circ h^{-1}$ defined on a sufficiently small half disk centered at the origin defines the real analytic parameterization of Σ_2 around q, while $\Pi_3^{-1} \circ h^{-1}$ defines that of Σ_3. Similarly, Σ_1 can be parameterized using the tangent plane $T_{p_0}(\Sigma_2)$. Denote these three maps parameterizing the neighborhood of p_0 by $u = (u_1,u_2,u_3) : Y_0 \to (\cup_{i=1}^3 \Sigma_i) \cup \gamma$ with $u_i : \Delta_i^+ \cup A_i \to \Sigma_i$. The real analyticity and the continuity of H follows from the construction. Q.E.D.

3 Construction of isothermal coordinates

Theorem 2 *Let $\Gamma \subset \mathbf{R}^3$ be a real-analytic curve and $\Sigma_i \subset \mathbf{R}^3$ $(i = 1,2,3)$ be a real-analytic surface so that $\Sigma_i \cup \Gamma$ is a real-analytic surface with boundary Γ. We assume further that the three surfaces are balanced, namely meeting along Γ at $120°$ degree angle. Then for every $p_0 \in \Gamma$, there exists an isothermal coordinate system of a neighborhood of p_0 by a conformal map from (Y_0, G_0) where G_0 is the triplet of standard Euclidean metric ${}^iG_0$ on each face Δ_i.*

PROOF. By Lemma 1, there exists a parameterization of a neighborhood of p_0 in the singular surface so that u_i is real analytic in $\Delta_i^+ \cup A_i$. Let ${}^iG$ be the the pull back of the Euclidean metric on $\mathbf{R}^n$ under the map u_i. With respect to the Euclidean coordinates of Δ_i, denote the metric components of ${}^iG$ by ${}^iG_{\alpha\beta}$. We now wish to find an isothermal coordinate by solving the Beltrami equation

$$w_{\bar{z}} = \mu w_z$$

where Beltrami coefficient μ is given by

$$\mu = \frac{{}^iG_{11} - {}^iG_{22} + 2\sqrt{-1}\,{}^iG_{12}}{{}^iG_{11} + {}^iG_{22} + 2\sqrt{{}^iG_{11}\,{}^iG_{22} - {}^iG_{12}^2}}.$$

Note here that the Beltrami coefficient μ is represented by the pull-back metric ${}^iG = (u_i)^*G_0$. The Beltrami coefficient μ has moduli strictly less than one. Furthermore, the metric components ${}^iG_{\alpha\beta}$ are given by

$${}^iG_{\alpha\beta} = \langle \frac{\partial u_i}{\partial x^\alpha}, \frac{\partial u_i}{\partial x^\beta} \rangle_{\mathbf{R}^n}.$$

Thus, the components of ${}^iG$ are real analytic on $\Delta_i^+ \cup A_i$ since the map u_i is real analytic there. Note that the quantity $\sqrt{{}^iG_{11}\,{}^iG_{22} - {}^iG_{12}^2}$ is the pulled-back area form of the immersed surface $u_i(\Delta_i^+ \cup A_i)$ in $\mathbf{R}^n$ by a real analytic map u_i. As the differential of the map u_i is non-degenerate by construction, the term ${}^iG_{11}\,{}^iG_{22} -{}^i G_{12}^2$ is strictly positive. Then the quantity $\sqrt{{}^iG_{11}\,{}^iG_{22} - {}^iG_{12}^2}$

is real analytic. $(G_{11} - G_{22} + 2\sqrt{-1}G_{12})(x,y)$ and $(G_{11} + G_{22} + 2\sqrt{G_{11}G_{22} - G_{12}^2})(x,y)$ on the open set R near the origin.

Let $P = u_i(p)$ be a point on the free boundary with $p = (x_0, 0) \in R \cap A$. We set $w(z,\bar{z}) = \alpha(x,y) + i\beta(x,y)$ and $\mu = \eta(x,y) + i\zeta(x,y)$ to rewrite the Beltrami equation defined on the half disk $\triangle_i^+$ as the following system of equations with real analytic coefficients:

$$\begin{pmatrix} \alpha_y \\ \beta_y \end{pmatrix} = \begin{pmatrix} \zeta & (1+\eta) \\ (1-\eta) & \zeta \end{pmatrix}^{-1} \begin{pmatrix} (1-\eta)u_x + \zeta v_x \\ -\zeta u_x - (1+\eta)v_x \end{pmatrix}$$

The inverse matrix on the right hand side exists because $|\mu|^2 = \eta^2 + \zeta^2 < 1$. We also have the Cauchy initial data

$$\alpha(x,0) = x - x_0 \text{ and } \beta(x,0) = 0$$

for $(x,0) \in A_i$ near $(x_0,0) \in A_i$. Therefore, we can apply the Cauchy-Kowalewski Theorem and obtain, in some neighborhood of the point p, a unique solution to the Beltrami equation. This solution w_i is a quasiconformal diffeomorphism from a neighborhood $\mathcal{U}_i \subset \triangle_i^+ \cup A_i$ of $(x_0,0)$ to a neighborhood $\mathcal{V}_i \subset \triangle_i^+ \cup A_i$ of $(0,0)$. By construction, the pulled-back metric of the Euclidean metric G_0 of $\triangle_i^+$ under w_i is conformal to ${}^iG$. Thus the map w_i^{-1} provides a parameterization of the neighborhood of $p = (x_0,0)$ in $(\triangle_i^+ \cup A_i, {}^iG)$ by an open set $\mathcal{V}_i$ in $\triangle_i^+ \cup A_i$. After scaling, we have constructed an isothermal coordinate system $F = (F_1, F_2, F_3) : (Y_0, G_0) \to (Y_0, G)$ of $p \in A$. The map $f = (f_1, f_2, f_3)$ with $f_i = u_i \circ F_i$ satisfies the desired properties of the isothermal coordinate system of $P \in \Gamma$. Q.E.D.

4 Harmonic functions on simplicial complexes

Let $f = (f_1, f_2, f_3) : (Y_0, G_0) \to \mathbf{R}^3$ be as in Theorem 2. The equality

$$f_i(x,0) = f_j(x,0) \tag{1}$$

for $i,j = 1,2,3$ implies

$$\frac{\partial f_i}{\partial x}(x,0) = \frac{\partial f_j}{\partial x}(x,0). \tag{2}$$

Using the conformality of f_i, the balancing of the three surfaces along the singular curve Γ can be written as

$$0 = \sum_{i=1}^{3} \frac{\frac{\partial f_i}{\partial y}}{|\frac{\partial f_i}{\partial y}|}(x,0) = \sum_{i=1}^{3} \frac{\frac{\partial f_i}{\partial y}}{|\frac{\partial f_i}{\partial x}|}(x,0).$$

This combined with (2) implies

$$0 = \sum_{i=1}^{3} \frac{\partial f_i}{\partial y}(x,0). \tag{3}$$

If we let

$$\tilde{f}_1(x,y) = -f_1(x,-y) + \frac{2}{3}\sum_{i=1}^{3} f_i(x,-y) \tag{4}$$

then (1) and (3) imply that

$$f_1(x,0) = \tilde{f}_1(x,0) \quad \text{and} \quad \frac{\partial f_1}{\partial y}(x,0) = \frac{\partial \tilde{f}_1}{\partial y}(x,0). \tag{5}$$

We claim (5) shows $U_1 : \triangle \to \mathbf{R}^n$ defined by setting

$$U_1(x,y) = \begin{cases} f_1(x,y) & \text{for } y \geq 0 \\ \tilde{f}_1(x,y) & \text{for } y < 0 \end{cases}$$

is harmonic. Indeed, for any smooth $\xi : \triangle \to \mathbf{R}^n$ with compact support, integration by parts gives

$$-\int_{\triangle^+} \nabla\xi \cdot \nabla U_1 dxdy = \int_{\triangle^+} \xi \triangle f_1 dxdy - \int_I \xi \frac{\partial f_1}{\partial y}(x,0)dx$$

and

$$-\int_{\triangle^-} \nabla\xi \cdot \nabla U_1 dxdy = \int_{\triangle^-} \xi \triangle \tilde{f}_1 dxdy + \int_I \xi \frac{\partial \tilde{f}_1}{\partial y}(x,0)dt$$

where $\triangle^+ = \{(x,y) \in \triangle : y > 0\}$, $\triangle^- = \{(x,y) \in \triangle : y < 0\}$ and $I = \{(x,y) \in \partial\triangle^+ : y = 0\}$. Summing up the above two equations and using the harmonicity of f_1 and $\tilde{f}_1$, we obtain

$$-\int_{\triangle} \nabla\xi \cdot \nabla U_1 dxdy = 0.$$

By Weyl's Lemma, U_1 is a C^ω harmonic map. Similarly, there exists C^ω extensions U_2, U_3 of f_2 and f_3. We call this construction of the real analytic extension U_i of f_i the *multi-sheeted reflection.*

By summarizing the argument above, we have

Theorem 3 *Let Σ_1, Σ_2, Σ_3 and γ as in Theorem 2. The surface Σ_i can be extended real analytically across the curve γ. This extended surface is parametrized by the conformal, harmonic map U_i via the multi-sheeted reflection.*

We note that the extendability of the minimal surface Σ_i across a real analytic boundary curve γ follows from a celebrated result of H.Lewy [Le]. On the other hand, Theorem 3 gives a more precise picture of the extension. Indeed, the extension of the parameterization f_1 of Σ_1 is given in terms of a linear combination of odd reflections of f_1, f_2, f_3 as defined in (4).

When three minimal surfaces are geometrically balanced along a C^ω curve in $\mathbf{R}^3$ (i.e. the unit outer normal of the three surfaces sum to zero as in (3)), the entire configuration is completely determined by one of the three surfaces. This follows from the so-called Bjöling's problem resolved by H.Schwarz. We will explain below how to use this and arguments in the proof of Theorem 3 to give a construction of Lewy's extension.

Theorem 4 *A minimal surface in $\mathbf{R}^3$ with a real analytic boundary can be extended across the boundary by a multi-sheeted reflection.*

PROOF. We start with a surface Σ_1 with a real analytic boundary curve γ. Let η_1 be the unit outer normal to the surface Σ_1 along γ. Let η_2 and η_3 be the two unit vector fields defined on γ, normal to γ, each making the angle of $\pi/3$ to η_1, Note here $\eta_1 + \eta_2 + \eta_3 = 0$. The solution by Schwarz of the Björlng's problem ([Ni] III §149) then provides locally defined, uniquely determined, minimal surfaces Σ_2 and Σ_3 along γ so that η_2 and η_3 are unit outer normals to Σ_2 and Σ_3 along γ respectively.

Recall that we have the harmonic and conformal parameterization $f_i : \Delta_i^+ \cup A_i \to (\Sigma_i \cup \Gamma) \subset \mathbf{R}^3$ without branch point. Furthermore, we also have

$$\sum_{i=1}^{3} \frac{\partial f_i}{\partial y} = c \sum_{i=1}^{3} \eta_i = 0$$

where $c = \left|\frac{\partial f_i}{\partial x}\right| = \left|\frac{\partial f_i}{\partial y}\right|$. Now each $f_i : \Delta_i \to \mathbf{R}^3$ can be extended across the real axis A_i by $\tilde{f}_1$ of (4). In particular, we have a conformal parameterization of the extension of Σ_1 as in Theorem 3. Q.E.D.

Recall that on a locally finite simplicial complex of dimension l, a function f is called *harmonic* if for every simplex σ of dimension $l-1$, the average value of the function f on all maximal simplices whose closure contain σ is zero. We demonstrate that the map which provides our local uniformization by Y_0 is harmonic in this sense after a normalization. In particular, we show that

Theorem 5 *The coordinate functions of the map $f : Y_0 \to \mathbf{R}^3$ of the singular minimal surface satisfy the mean value equality:*

$$\int_{B_\varepsilon(p_0)} \sum_{i=1}^{3} [f_i(x) - f(p_0)] \; dx = 0$$

where $p_0 = (x_0, 0)$ in A, and $B_\varepsilon(p_0)$ is a ball of radius $\varepsilon > 0$ in (Y_0, G_0), namely the set $\{y \in Y_0 | d(y, p_0) < \varepsilon\}$ where the distance function d is with respect to the Euclidean metric ${}^iG_0$ on each face Δ_i^+. Here we note $f_i(p_0) = f(p_0)$ for $i = 1, 2, 3$.

PROOF. We have shown above that each map f_i defined on Δ_i^+ can be canonically extended across the edge A to the disc Δ_i. The resulting harmonic function U_i satisfies the mean value equality

$$\frac{1}{\pi\varepsilon^2} \int_{B_\varepsilon(p_0)} U_i(x) dx = f_i(p_0).$$

Rewriting the integral as a sum of integrals over the upper disc $B_\varepsilon^+(p_0) \subset \Delta_i^+$ and the lower disc $B_\varepsilon^-(p_0) \subset \Delta_i^-$, we get

$$\frac{1}{\pi\varepsilon^2/2} \int_{B_\varepsilon^+(p_0)} f_i(x) dx = -\frac{1}{\pi\varepsilon^2/2} \int_{B_\varepsilon^-(p_0)} \tilde{f}_i(x) dx + 2 f_i(p_0).$$

By rewriting $\tilde{f}_i(x, y)$ as $-f_i(x, -y) + \frac{2}{3} \sum_{j=1}^{3} f_j(x, -y)$ for $y < 0$, and taking a sum over $i = 1, 2, 3$, we get

$$\sum_{i=1}^{3} \frac{1}{\pi\varepsilon^2/2} \int_{B_\varepsilon^+(p_0)} f_i(x) dx = 3 f(p_0),$$

which is the mean value equality. Q.E.D.

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